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PROFILE OF A CYLINDRICAL SHOCK WAVE AND THE "PEAK" APPROXIMATION*

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A solution of the problem of the propagation of an axisymmetric shock wave formed as a result of the initial velocity jump is obtained. The solution yields a universal characteristic profile of cylindrical shock waves in the range of small Mach numbers. In the case of a gas, this range corresponds to weak sonic perturbations, while in the case of liquids the framework of the approximation given can be used to study underwater explosions and intense electric discharges.

The heuristic peak approximation is often used in computing spherical, as well as cylindrical shock waves. According to this approximation the pressure drop in the cavity formed as a result of an explosion has an exponential time dependence /1, 2/. This relationship enables us to obtain, e.g. in the framework of the Kirkwood-Bethe theory, an exponential distribution of the parameters in the normal zone of increased pressure propagating directly behind the shock (the "profile" of the shock wave /1/).

An exponential profile was obtained in /3/ for spherical shock waves without any additional assumptions, on the basis of an analytic solution of the initial boundary value problem of a spherical piston beginning to move with non-zero initial velocity. The solution proves the presence of a characteristic profile of spherical shock waves in the range of Mach numbers under consideration, unconnected with any specific law governing the decrease in pressure inside the cavity, and agrees well /4/ with experimental results. Below, a non-exponential profile is obtained from cylindrical shock waves.

The asymptotic method used here was employed earlier in solving the problem of a smoothly moving piston /5, 6/, in the study of the capsizing of a spherical compression wave /7/ and in a number of other problems.

1. Formulation of the problem. Asymptotic solution. The motion of a perfect medium is described, in the case of spherical symmetry, by the set of equations

$$u_t + uu_r = -\frac{1}{\rho}\rho_r c^2, \quad \rho_t + u\rho_r + \rho\left(u_r + \frac{u}{r}\right) = 0 \tag{1.1}$$

where the pressure and density are connected by the barotropy condition, and the speed of sound $c = \sqrt{p_\rho}$. The barotropy condition replaces the energy equation alsc an the shock wave.

Let a cylindrical axisymmetric cavity of radius r_0 form at the initial instant, whose boundary henceforth will play the part of the piston moving according to the law $r = R(t)$. A characteristic feature of shock-type problems is the presence of a discontinuity in the initial parameters of the medium caused by the non-zero initial velocity of the piston $R'(0) = U$.

Let us introduce the dimensionless variables

$$x = \frac{r}{r_0}, \quad v = \frac{u}{U}, \quad \tau = \frac{tU}{r_0}, \quad q = \left(\frac{\rho}{\rho_0} - 1\right)\frac{c_0^2}{U^2}, \quad M = \frac{U}{c_0}$$

where ρ_0, c_0 are the density and speed of sound in the unperturbed medium. In dimensionless variables system (1.1) becomes

$$\begin{aligned} &v_\tau + vv_x = -\frac{1}{1 + qM^2} q_x \left(\frac{c}{c_0}\right)^2 \\ &M^2 q_\tau + M^2 v q_x + (1 + qM^2)\left(v_x + \frac{v}{x}\right) = 0. \end{aligned} \tag{1.2}$$

We shall seek an asymptotic solution to the problem formulated above, regarding the Mach number of the unperturbed medium as a small parameter, under the condition $R'(t)/U \sim 1$.

Using the condition of barotropy when $qM^2 \ll 1$, we can carry out the following expansion in the equation of motion:

$$\frac{(c/c_0)^2}{1 + qM^2} = 1 + M^2 qk\,.$$

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The condition on the piston takes the form $x = \varphi(\tau)$ and $\varphi(0) = 1$, $d\varphi/d\tau|_0 = 1$. The conditions on the weak shock wave are

$$\frac{dx_s}{d\tau} = \frac{1}{M} + \frac{v_s}{2}, \quad v_s = q_s M. \tag{1.3}$$

To construct the asymptotic solution, we will introduce two zones, construct the expansions in these zones and use the asymptotic matching method to find the arbitrary functions appearing in the expansions. The first zone is characterized by short times, and the scales of the variables in it are determined by introducing new variables of the order of unity in the corresponding zone

$$v = v^c, \; q = q^c/M, \; \tau = \tau^c M, \; x = x^c. \tag{1.4}$$

In the second zone the scales of the variables are determined by the expressions

$$v = v^i \sqrt{M}, \; q = q^i/\sqrt{M}, \; \tau = \tau^i, \; x = x_i/M. \tag{1.5}$$

This zone corresponds to times of the order of unity, and large distances. The solution in the zone (1.5) can be found by matching with the solution in zone (1.4), which can be found from the conditions on the piston and on the shock wave.

The principal term of the expansion in zone (1.4) is described by the system of equations

$$\frac{\partial v^c}{\partial \tau^c} = -\frac{\partial q^c}{\partial x^c}, \quad \frac{\partial q^c}{\partial \tau^c} + \frac{\partial v^c}{\partial x^c} + \frac{v^c}{x^c} = 0. \tag{1.6}$$

The times in the zone (1.4) are short, and therefore the relation $R(\tau^c M) = 1$ holds for the piston velocity in the principal approximation. The condition on the piston in the principal approximation is displaced in the zone (1.4) to the line $x^c = 1$, i.e.

$$v^c(1, \tau^c) \equiv 1. \tag{1.7}$$

Simplifying this condition produces a universal characteristic profile of the shock waves, since it only contains the asymptotic form of the law of motion of the piston written in dimensionless coordinates, in universal form.

Condition (1.3) in the principal approximation, on the shock wave, is displaced in zone (1.4) to the characteristic of the system (1.6) : $x^c = 1 + \tau^c$, and matches the condition on the characteristic, Solving the ordinary differential equation describing the distribution of the parameters along the characteristic, we can obtain the condition on the shock wave in the form

$$v^c = q^c = 1/\sqrt{x^c}, \; x^c = x = 1 + \tau^c. \tag{1.8}$$

Hence, we obtain the characteristic Cauchy problem in zone (1.4), with the data on the characteristic. In the simpler case of spherical symmetry, the equations have a solution in the form of travelling waves, although such a simple solution is not possible in the case of cylindrical symmetry. Below we use a method suitable for both cases of symmetry.

2. Constructing a solution in a short time zone in the form of series in terms of the characteristic variable. We will seek a solution of the problem (1.6)–(1.8) in the form of series in terms of the characteristic variable $\xi = 1 - x + \tau^c$

$$v^c = \sum_{n=0}^{\infty} v_n^c(x)\xi^n, \quad q^c = \sum_{n=0}^{\infty} q_n^c(x)\xi^n. \tag{2.1}$$

Substituting these relations into system (1.6) we observe that the coefficients of the series (2.1) must be sought in the form

$$v_n^c(x) = \frac{1}{\sqrt{x}} \sum_{j=0}^{\infty} \beta_j^n x^{-j}, \quad q_n^c = \frac{1}{\sqrt{x}} \sum_{j=0}^{\infty} \alpha_j^n x^{-j}. \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.6), we obtain

$$\beta_j^k = -\alpha_j^k \frac{2j+1}{2j-1} \tag{2.3}$$

$$\alpha_j^k = -\frac{(2j-1)^2}{8jk} \alpha_{j-1}^{k-1}. \tag{2.4}$$

The recurrent formula (2.4) yields coefficients with large k, provided that the coefficients of the preceding step are known. The initial coefficients are found from condition (1.8) on the characteristic $\xi = 0$

$$\sum_{j=0}^{\infty} \beta_j^{\circ} x^{-j} = \sum_{j=0}^{\infty} \alpha_j^{\circ} x^{-j} \equiv 1$$

from which we obtain $\beta_0^{\circ} = \alpha_0^{\circ} = 1,\ \beta_j^{\circ} = \alpha_j^{\circ} = 0\ (j > 0)$. Therefore from (2.4) it follows that $\alpha_j^k = \beta_j^k = 0$ when $j > k$.

Conditions (1.7) on the piston yield

$$\sum_{j=0}^{\infty} \sum_{k=0}^{\infty} \beta_j^k \xi^k = 1, \quad \sum_{j=0}^{\infty} \beta_j^{\circ} = 1, \quad \sum_{j=0}^{\infty} \beta_j^k = 0 \quad (k > 0)\,. \tag{2.5}$$

The condition on the piston matches the condition on the characteristic; therefore conditions (2.3) and (2.5) are also matched.

To express α_0^k in terms of the values from the preceding step, we use relations (2.5) to obtain

$$\alpha_0^k = \beta_0^k = \sum_{j=1}^{k} \alpha_j^k \frac{2j+1}{2j-1}\,. \tag{2.6}$$

Formula (2.5) yields

$$\alpha_0^k = \frac{4}{k!} \sum_{n=0}^{k-1} \frac{1}{(-32)^{n-1}} \frac{2n+3}{2n+1} \frac{[(2n+1)!]^2}{(n!)^2} \frac{(k-n-1)!}{(n+1)!} \alpha_0^{k-n-1}\,. \tag{2.7}$$

Expressing α_j^k in terms of the previous coefficients using the recurrence formula, we arrive either at the coefficients α_j°, or at α_0^k. The first of these coefficients for which formula (2.7) holds, are $\alpha_j^{\circ} = \delta_j^{\circ}$. We can also obtain from (2.4)

$$\alpha_j^k = -\frac{1}{8^j} \frac{[(2j-1)!!]^2 (k-j)!}{j!\,k!} \alpha_0^{k-j} \quad (j \leqslant k)\,. \tag{2.8}$$

Hence, we have found all the coefficients.

We shall show that series (2.1) converge for all x in the region of flow when $\xi < 2$.

First we will show by induction that

$$|\alpha_0^k| \leqslant 2^{-k}\,. \tag{2.9}$$

We shall assume that condition (2.9) holds for all $i < k$ and prove it for $i = k$. Let us rewrite formula (2.7) in the form

$$\alpha_0^k = \sum_{n=0}^{k-1} \frac{2n+3}{2n+1} (-{}^1/_2)^{n+1} \frac{(n+{}^1/_2)^2 (n-{}^1/_2)^2 \ldots ({}^3/_2)^2 ({}^1/_2)^2}{(n-1)\,n \ldots 2 \cdot 1} \frac{\alpha_0^{k-n-1}}{k\,(k-1) \ldots (k-n)}\,. \tag{2.10}$$

Let us estimate $|\alpha_0^k|$. To do this we separate in (2.10) the term corresponding to $n = 0$ and write for the remaining terms $(2n+3)/(2n+1) \leqslant {}^5/_3$. Then

$$|\alpha_0^k| \leqslant \frac{3}{8k} |\alpha_0^{k-1}| + \frac{1}{8} \cdot \frac{5}{3} \sum_{n=1}^{k-1} \frac{1}{2^n} \frac{(n+{}^1/_2)(n-{}^1/_2) \ldots {}^3/_2 \cdot {}^1/_2 \cdot {}^3/_4}{k\,(k-1) \ldots (k-n)} |\alpha_0^{k-n-1}| \leqslant \frac{3}{8k} |\alpha_0^{k-1}| + \frac{5}{32} \sum_{n=1}^{k-1} \frac{|\alpha_0^{k-n-1}|}{2^n C_k^{n+1}}$$

where C_k^n is the number of combinations of k elements taken n at a time.

The assumption of the induction method implies the estimate

$$|\alpha_0^k| \leqslant \left(\frac{3}{4k} + \frac{5}{16} \sum_{n=1}^{k-1} \frac{1}{C_k^{n+1}} \right) \cdot 2^{-k} \tag{2.11}$$

and we have the following estimate for the sum:

$$\sum_{n=1}^{k-1} \frac{1}{C_k^{n+1}} = 1 + \sum_{n=1}^{k-2} \frac{1}{C_k^{n+1}} \leqslant \frac{2k+2}{k} \tag{2.12}$$

since $C_k^{n+1} \geqslant k$ when $1 \leqslant n \leqslant k-2$.

Taking the estimate (2.12) into account we obtain from (2.11) for $k \geqslant 4$ inequality (2.9). The inequality can be confirmed directly for $k = 0, 1, 2, 3$

$$\alpha_0^{\circ} = 1, \quad \alpha_0^1 = -\frac{3}{8}, \quad \alpha_0^2 = \frac{33}{256}, \quad \alpha_0^3 = -\frac{83}{2048}$$

and this proves the condition (2.9).

From (2.8), (2.9) we can obtain the estimate for the remaining coefficients

$$|\alpha_j^k| \leqslant |\alpha_0^{k-j}| \frac{(2j-1)!!}{4^j k(k-1)\ldots(k-j+1)} \leqslant |\alpha_0^{k-j}| \frac{j!(k-j)!}{2^j k!} = |\alpha_0^{k-j}| \cdot 2^{-j}/C_k^{\,j} \leqslant 2^{1-k}/C_k^{\,j}.$$

This yields the following expression for the coefficients of the second series of (2.1):

$$|q_k^{\circ}(x)| = \sum_{j=0}^{k} |\alpha_j^{\,k}|\, |x^{-j}| \leqslant 2^{1-k} \sum_{j=0}^{k} \frac{1}{C_k^{\,j}} \leqslant A \cdot 2^{-k}$$

since $x \geqslant 1$ in the region of flow and the numbers

$$\sum_{j=0}^{k} \frac{1}{C_k^{\,j}}$$

are bounded uniformly in k.

Consequently the series (2.1) for the pressure certainly converges when $x \geqslant 1,\ 0 \leqslant \xi \leqslant 2$.

From (2.3) it follows that $|\beta_j^k| \leqslant 4|\alpha_j^k|$. This clearly implies that series (2.1) for the velocity also converges when $x \geqslant 1,\ 0 \leqslant \xi \leqslant 2$.

3. Pressure and velocity profile of a cylindrical shock wave. Using the solution obtained we find the expansion in the zone (1.5) determined by the time intervals of the order of unity. Howerver, since in the case of the zone (1.4) the solution is suitable up to $\xi = 2$ (its further suitability has not been proved), we shall construct the solution not over the whole zone (1.5), but within its subregion N (Fig.1) specified by the condition $\xi = 1 - x + \tau/M \sim 1$. It is precisely this zone into which the perturbations are carried from zone (1.4) by the characteristics with a positive direction.

Let us replace, in the zone in question, the coordinate τ_i, x_i by ξ, x_i. System (1.2) will now become

$$\frac{1}{\sqrt{M}} \frac{\partial v^i}{\partial \xi} - M v^i \left(M \frac{\partial v^i}{\partial x_i} - \frac{\partial v^i}{\partial \xi} \right) = - \frac{\sqrt{M}}{1 + q_i M^{3/2}} \frac{\partial q^i}{\partial x_i} - \frac{1}{\sqrt{M}} \frac{\partial q^i}{\partial \xi} \tag{3.1}$$

$$\sqrt{M} \frac{\partial q^i}{\partial \xi} + M^2 v^i \left(M \frac{\partial q^i}{\partial x_i} - \frac{\partial q^i}{\partial \xi} \right) + \tag{3.2}$$

$$(1 + q^i M^{3/2}) \sqrt{M} \left(M \frac{\partial v^i}{\partial x_i} - \frac{\partial v^i}{\partial \xi} + \frac{v^i}{x_i} M \right) = 0.$$

Both these equations yield, in the principal approximation, the same equation $v_\xi^i = q_\xi^i$ which, taking into account the conditions on the shock wave (1.3), reduces to

$$v^i = q^i. \tag{3.3}$$

We find, however, that the condition of matching the subsequent terms of the expansion yields another equation connecting the principal terms of the expansion.

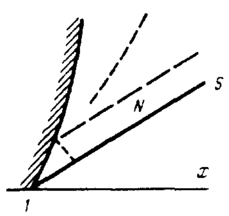

Fig.1

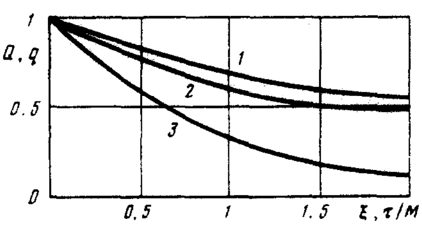

Fig.2

The condition can be obtained by multiplying (3.1) additionally by M, and subtracting it from (3.2). This yields, in the principal approximation,

$$\frac{\partial v^i}{\partial x_i} + \frac{v^i}{x_i} = - \frac{\partial q^i}{\partial x_i}. \tag{3.4}$$

From (3.3) and (3.4) we obtain

$$v^i = q^i = \Omega(\xi)/\sqrt{x_i}. \tag{3.5}$$

Matching (3.5) with (2.1) we find $\Omega(\xi)$. To do this we introduce the auxilliary coordinates $x_1 = x\sqrt{M},\ \xi_1 = \xi$. The principal term of solution (2.1) will be written in auxilliary coordinates in the form

$$v = \sqrt[4]{M} \sum_{n=0}^{\infty} \frac{\beta_0^n \xi_1^n}{\sqrt{x_1}} .$$

This shows that matching takes place when

$$\Omega = \sum_{n=0}^{\infty} \beta_0^n \xi^n .$$

We note that matching the densities would give the same result, since by virtue of (2.3) $\beta_0^n = \alpha_0^n$.

Finally, the density and velocity profile will take the following form in the zone near the shock wave, when the times are of the order of unity:

$$q = \frac{v}{M} = \frac{1}{M\sqrt{x}} \sum_{n=0}^{\infty} \alpha_0^n \xi^n . \tag{3.6}$$

As was shown above, the radius of convergence of this series is not less than two.

4. Comparison with a spherical shock wave and the "peak" approximation.

Using the same asymptotic conditions as in the present paper, an expression was obtained in /3/ for the profile of a spherical shock wave for an arbitrary law of motion in the form

$$q = v/M = e^{-\xi}/(Mx). \tag{4.1}$$

If we disregard the difference in the fall of the amplitude between the cylindrical and spherical shock wave, which is a well-known fact, we see from (3.6) that in the case of cylindrical symmetry the characteristic profile does not obey an exponential relationship and for this reason the peak approximation gives an incorrect wave profile. Expanding the exponential term in (4.1) in a series, we obtain the following profile for the sperhical wave:

$$Q = 1 - \xi + \xi^2/2 - \xi^3/6 + \ldots \tag{4.2}$$

while the profile of the cylindrical shock wave has the form

$$Q = \sum_{n=0}^{\infty} \alpha_0^n \xi^n = 1 - \frac{3}{8}\xi + \frac{33}{256}\xi^2 - \frac{83}{2048}\xi^3 + \frac{1971}{262144}\xi^4 - \ldots \tag{4.3}$$

We note that (2.7) implies that the series (4.3) has alternating signs; therefore the error can be easily estimated by restricting ourselves to several terms.

At a fixed instant of time the variable ξ will represent the distance between the leading perturbation front corresponding to $\xi = 0$, and the point lying at a distance x from the centre of symmetry. For this reason formula (3.6) describes the profile of the shock wave in terms of a length equal to two initial radii of the piston. According to the formulas (3.6), (4.1) obtained the width of the shock wave does not increase with time, although experiments show some increase caused by dissipation. At distance equal to two initial radii the peak pressure falls almost tenfold in the case of spherical symmetry, and by almost half in case of cylindrical symmetry. In Fig.2 using the variables Q, ξ the upper curve *1* shows the profile of a cylindrical shock wave, and curve *3* that of a spherical shock wave. In the spherical wave the pressure drops much more rapidly and its width is much smaller than in the case of cylindrical symmetry.

The results obtained can be interpreted somewhat differently, since the characteristic profiles (3.6), (4.1) are generated by the motion of the piston at the initial instant. Indeed, $\xi \sim 1$ corresponds to times $\tau \sim M$ in the zone (1.4). In the principal approximation we can assume that the boundary of the piston does not vary and is equal to unity in dimensionless coordinates. Therefore, we have $\xi = \tau/M$ at the piston in the zone (1.4). We have the following expressions for the pressure at the piston in the case of cylindrical and spherical symmetry respectively:

$$q = \frac{1}{M} \sum_{n=0}^{\infty} \sum_{j=0}^{n} \beta_j^n \left(\frac{\tau}{M}\right)^n = \frac{1}{M}\left[1 - \frac{1}{2}\frac{\tau}{M} + \frac{3}{16}\left(\frac{\tau}{M}\right)^2 - \frac{1}{16}\left(\frac{\tau}{M}\right)^3 + \ldots\right] \tag{4.4}$$

$$q = \frac{1}{M} e^{-\tau/M} \tag{4.5}$$

The series (4.4) converges when $\tau/M < 2$.

The lines *2* and *3* in Fig.2 in the variables $q, \tau/M$ represent the relations (4.4), (4.5). In the case of a spherical shock wave its profile (4.1) shows the same exponential relationship as the decrease in pressure with time at the piston (4.5). In this case the peak approximation for the pressure in the cavity is confirmed theoretically. In the case of cylindrical symmetry the wave profile (curve *1*) differs from the plot showing the dependence of the pressure at the piston on time (curve *2*) by 15—20% on average. The law governing the drop in pressure in a cavity differs considerably in the axisymmetric case from the exponential relation (4.5).

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DIFFRACTION OF A SINGLE PLANE WAVE BY A V-SHAPED WING*

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A linear formulation is used to solve the problem of the diffraction of a single plane wave by a *V*-shaped wing moving at supersonic speed. The solution is based on the study of the eigenfunctions for a class of selfsimilar solutions of the three-dimensional wave equation. The boundary integral is constructed using a method analogous to that discussed in /1/, and results obtained in /1—3/ are used.

1. We shall seek a solution of the wave equation

$$\frac{\partial^2\Phi}{\partial x^2}+\frac{\partial^2\Phi}{\partial y^2}+\frac{\partial^2\Phi}{\partial z^2}-\frac{\partial^2\Phi}{\partial t^2}=0 \tag{1.1}$$

for the homogeneous functions of zero dimensions in t and $q=(x^2+y^2+z^2)^{1/2}$.

It was shown in /1/ that knowing the homogeneous solution of zero dimensions and using the relation

$$\Phi_n=\frac{(-1)^n(t^2-q^2)^{n+1}}{2^n n!}\frac{\partial^n}{\partial t^n}\left(\frac{\Phi_0}{t^2-q^2}\right)$$

we can obtain a uniform solution of dimensions $n\in N$. Here Φ_0 and Φ_n are solutions of (1.1) uniform in t and q, of dimensions zero and n respectively.

We have the following representation for the uniform solution Φ_0 of zero dimensions in the form of a series in eigenfunctions:

$$\Phi_0=-(\rho^2-1)\frac{\partial}{\partial\rho}\sum_{n=0}^{\infty}\Big[\frac{A_{n,0}}{2}G_{n,0}(\varphi)Q_n(\rho)+$$

$$\sum_{k=1}^{\infty}(A_{n,k}\cos k\lambda\theta+B_{n,k}\sin k\lambda\theta)G_{n,k}(\varphi)Q_{n+k\lambda}(\rho)\Big]$$

$$G_{n,k}^{(\varphi)}=C_n^{k\lambda+1/2}(\cos\varphi)\sin^{k\lambda}\varphi$$

$$\left.\begin{matrix}A_{n,k}\\B_{n,k}\end{matrix}\right\}=\frac{2n!\,(n+k\lambda+1/2)\,\Gamma(k\lambda+1/2)\,\Gamma(2k\lambda+1)}{T\sqrt{\pi}\,\Gamma(n+2k\lambda+1)\,\Gamma(k\lambda+1)}\times$$

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